A field-theoretical approach to the spin glass transition: models with long but finite interaction range

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Abstract

We study spin glasses with Kac type interaction potential for small but finite inverse interaction range γ. Using the theoretical setup of coupled replicas, through the replica method we argue that the probability of overlap profiles can be expressed for small γ through a large-deviation functional. This result is supported by rigorous arguments, showing that the large-deviation functional provides at least upper bounds for the probability. Finally we analyze the rate function, in the vicinity of the critical point $T_c = 1$, $h = 0$ of mean field theory, and we study the free energy cost of overlap interfaces, assuming the validity of a gradient expansion for the rate functional.

1 Introduction

In recent times the introduction of interpolating techniques by Guerra [1], and their smart generalization by Talagrand [2] to the case of two coupled replicas of the system, has led to the much awaited proof of the Parisi Ansatz [3] for mean field spin glasses. The nature of the glassy phase of disordered systems in finite dimension remains on the other hand a largely open problem.

The question is relevant both for the spin-glass problem, modeled by Hamiltonians with disordered two-body interactions, and for the structural glass problem, for which, despite the

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absence of intrinsic quenched disorder, a phenomenological analogy has been suggested with the so called p-spin models, i.e., mean-field spin-glasses with p-spin ($p > 2$) random interactions.

In both contexts, replica theory suggests that the mean field equilibrium phase diagram and the nature of the glassy phase should remain qualitatively unchanged above a finite critical dimension [4]. This description has been thwarted by phenomenological descriptions – droplet theories [5] – arguing that Replica Symmetry Breaking is unstable in any finite-dimensional space. Droplet theories have been comforted by mixtures of rigorous and heuristic arguments which, while clarifying specific aspects of finite-dimensional disordered systems, do not give a definite answer to the question of the validity of the mean field phase diagram in finite dimension. In three-dimensional systems it is unlikely that the question can be solved experimentally or by numerical simulations. The dynamics of experimental spin glasses displays features such as a strong aging effect [6] and a non-trivial fluctuation-dissipation ratio [7] as predicted by mean field theory [8]. However, the observed dynamical regime is too far from the asymptotic situation that would allow to infer properties of the equilibrium phase [9]. In simulations, on the other hand, while dynamics suffers from the same problems as the experiments, the possibility of studying equilibrated samples is restricted to small systems and it cannot be excluded that the observed mean-field like behavior is dominated by finite-size effects. In this situation it is natural to look at models which, interpolating between mean field and finite range, allow for analytical treatment. The utility of such models would not be confined to the study of the equilibrium phase. Neglection of space in the mean-field dynamics of the p-spin model, for example, predicts an artifactual dynamical breaking of ergodicity that blocks the system to free-energy densities higher then the equilibrium one, realizing the scenario predicted by the idealized Mode-Coupling Theory for structural glasses [10]. The inclusion of finite-dimensional effects is necessary to study the dynamical processes that restore ergodicity allowing to go beyond Mode-Coupling Theory.

In recent work [11] we showed that, considering spin glasses with variable range of interaction γ^{-1} and taking the Kac limit of vanishing γ, one recovers the mean field free energy, thus generalizing the celebrated Lebowitz-Penrose theorem [12]. In addition to the free energy, the same convergence result applies to the local order parameter [13]. In other words, one can define a local overlap distribution function and show that this tends to the usual mean field overlap distribution function $P(q)$, which is non-trivial in the low temperature regime. Of course, this has no direct implication on the phase structure for small but finite γ, where one can expect a transition only for high enough dimensionality d. One can hope however that the $\gamma = 0$ limit can be taken as a starting point to study the spin glass transition in a simplified setting. This hope is comforted by what happens in non-disordered systems, e.g., the ferromagnetic Ising model. In that case, through a block-spin transformation one can introduce spatial magnetization profiles and express their probability by means of a suitable free-energy functional [14]. This leads to a field theory over the magnetization profiles, which can be analyzed "semiclassically", the role of the large parameter in front of the action being played by γ^{-d}. Thanks to that, customary low temperature techniques, of the kind of the Peierls argument or Pirogov-Sinai theory for Ising systems, can be adapted to prove the existence of a phase transition for small γ in dimension

large enough [22].

In this paper we address the problem of achieving an analogous field-theoretical representation for the spin glass problem. We consider a variant of the spin glass model with Kac interaction potential, the ultimate goal being that of studying the presence of long-range order for large but finite interaction range γ^{-1}. We first give a definition of long-range order as sensitiveness in the bulk to appropriately chosen boundary conditions. We apply to our problem the method of coupled replicas [15] and, as in [16], we consider two copies of the same system coupled in such a way to have local overlap fixed on the boundary. The appropriate field theory will then result from the evaluation of the free-energy cost for having overlap profiles on certain coarse-grained regions of space. In a first moment we resort to the replica method and use a generalization of the Parisi Ansatz suitable to study problems of coupled replicas [15], in order to evaluate the free-energy cost. Then, through the by now usual interpolating techniques [17], [1], [18] we prove that the replica expression provides a free-energy lower bound.

Once the field-theoretical representation has been obtained, one can envisage to use it to prove the existence of long-range order in high enough dimension. A necessary preliminary task, which we undertake in the present work, consists in the analysis of the properties of the rate function and in the evaluation of the cost of overlap profiles. This functional has a zero mode corresponding to flat profiles with overlap chosen in the support of the mean field $P(q)$ function. This implies that the lowest cost interfaces separating regions with homogeneous different values of the overlap will be extended in space.

Close to the critical point $T = 1$, $h = 0$ of mean-field theory, we give some interface estimates in the lines of [19], indicating that the cost for extended "overlap interfaces" grows with the linear dimension L of the system as $L^{d-5/2}/\gamma^{5/2}$. The main difficulty in inferring from this fact the presence of long-range order for $d > 5/2$ lies in the possible existence, that we cannot exclude, of entropic contributions scaling with higher powers of L, that would destroy the possibility of the phase transition for any $\gamma > 0$.

2 The model

The Hamiltonian of the Kac spin glass with two-body interactions, on the d-dimensional hypercube $\Lambda = \{1, \cdots, L\}^d$ and in presence of a magnetic field h, can be defined as

$$H_\Lambda(\sigma, h; J) = -\frac{1}{\sqrt{2}} \sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i. \tag{1}$$

The J_{ij} are Gaussian independent random variables with zero mean and variance

$$EJ_{ij}^2 = \gamma^d \phi(\gamma|i-j|), \tag{2}$$

where $\phi(|x|)$ is a smooth non-negative function with support in $|x| \leq 1$, and normalized so that

$$\int d^dx\, \phi(|x|) = 1. \tag{3}$$

We will also require ϕ to be non-negative definite, i.e.,

$$\int d^d x \, \phi(|x|) e^{ikx} \geq 0 \ \text{ for any } \ k \in \mathbb{R}^d. \tag{4}$$

In the theory of mean field spin glasses an important role is played by p-spin models where the spins are coupled through p-body interactions. It is interesting to generalize the definition of the Kac spin glass model to that case and define the Hamiltonian

$$H_\Lambda^{(p)}(\sigma, h; J) = -\frac{1}{\sqrt{2}} \sum_{i_1, \cdots, i_p \in \Lambda} J_{i_1 \cdots i_p} \sigma_{i_1} \cdots \sigma_{i_p} - h \sum_{i \in \Lambda} \sigma_i, \tag{5}$$

with

$$E(J^2_{i_1 \cdots i_p}) = \frac{\sum_{k \in \Lambda} \psi(\gamma |i_1 - k|) \cdots \psi(\gamma |i_p - k|)}{W(\gamma)^p} \tag{6}$$

and

$$W(\gamma) = \sum_{k \in \Lambda} \psi(\gamma |k|). \tag{7}$$

where $\psi(|x|)$, $x \in \mathbb{R}^d$, is any non-negative summable function with compact support, normalized as

$$\int d^d x \, \psi(|x|) = 1. \tag{8}$$

The rationale for the choice (6) among all possible functions of p variables with range γ^{-1} is discussed in [13] and ensures that for all β, h, Λ and for even p, the finite volume free-energy of the Kac model is bounded below by the mean field free-energy corresponding to the same external parameters and number of spins. Of course terms with different p can be combined. For these models we recently proved the convergence of the free energy to the mean field value in the Kac limit, provided that p is even, as well as the convergence of the distribution of the local order parameter [13].

For many purposes the analysis of the p-spin model is very similar to the one of the usual two-body Hamiltonian (1), while the notations for generic p are much heavier. For that reason we will present all our arguments in the case of the two-body model, and we will give at the end the general formulae in the p-spin case.

3 Overlap profiles and boundary conditions

At mean field level, the order parameter for the spin glass transition is the Parisi probability distribution, describing the statistics of the overlap between two replicas with the same disorder, induced by the Gibbs measure and by the disorder distribution. In Ref. [13] we showed that in Kac spin glasses the distribution of *local* overlaps, on scales of the order of the interaction

range, tends to the Parisi overlap probability distribution for $\gamma \to 0$, in any space dimension d. This does not tell anything about the possibility of long-range order where the probability distribution of the global overlap has a non-trivial shape. In order to discuss the possibility of long-range order, let us introduce the local overlaps, and the probability of local overlap profiles. First of all, let us partition $\mathbb{Z}^d$ into cubes Ω_k, $k \in \mathbb{Z}^d$, of side δ/γ (to be chosen to be an integer):

$$\Omega_k = \{i \in \mathbb{Z}^d \ : \ \frac{\delta}{\gamma} k_l \leq i_l < \frac{\delta}{\gamma}(k_l + 1); \ \ l = 1, ..., d\}. \tag{9}$$

The situation we have in mind is

$$1 \ll \frac{\delta}{\gamma} \ll \frac{1}{\gamma} \ll L \tag{10}$$

and, for definiteness, one can think that $\delta \sim \gamma^{1-\varepsilon}$ for some $0 < \varepsilon < 1$. Let us denote by $M = \left(L\frac{\gamma}{\delta}\right)^d$ the total number of boxes. Given two spin configurations σ^1 and σ^2 on Λ we define the local overlaps on the box Ω_k as

$$q_k(\sigma^1, \sigma^2) = \left(\frac{\gamma}{\delta}\right)^d \sum_{i \in \Omega_k} \sigma_i^1 \sigma_i^2 \tag{11}$$

and the global overlap

$$q(\sigma^1, \sigma^2) = \frac{1}{L^d} \sum_{i \in \Lambda} \sigma_i^2 \sigma_i^2 \tag{12}$$

Of course, as long as δ/γ is finite the local overlap can assume just a finite number of values. We define a "local overlap profile" $\{\tilde{p}_k\}_{k=1,\cdots,M}$ to be a collection of possible values for the local overlaps[1]. Given a realization J of the disorder we define the Boltzmann weight associated to a local overlap profile $\{\tilde{p}_k\}$ as

$$B_\Lambda[\{\tilde{p}_k\}; J] = \sum_{\sigma^1, \sigma^2} \exp(-\beta H_\Lambda(\sigma^1, h; J) - \beta H_\Lambda(\sigma^2, h; J)) 1_{\{q_k(\sigma^1, \sigma^2) = \tilde{p}_k \forall k\}}. \tag{13}$$

For fixed disorder, the probability of an overlap profile is just

$$P_\Lambda[\{\tilde{p}_k\}; J] = \frac{B_\Lambda[\{\tilde{p}_k\}; J]}{Z_\Lambda(\beta, h; J)^2}. \tag{14}$$

From the results of [13], one can argue that the functional $P_\Lambda[\{\tilde{p}_k\}; J]$ must be concentrated, in the situation (10), on profiles where each of the $\tilde{p}_k$ is in $Supp(P(q))$, the support of the mean field Parisi overlap distribution. The expectation from replica theory is that in high enough

[1] The definition we give here does not coincide exactly with the one adopted in [13], while its properties for small γ do not depend on the details of the definition

dimension and γ small enough, at temperatures smaller than $T_c = 1$, long-range order should be present. The two-replica free energy would be dominated by configurations with a constant overlap in space, their weight being given by the function $P(q)$.

The space homogeneity of typical overlap profiles can be argued considering that in any fixed subset $\Omega \subset \Lambda$ containing an extensive number of spins $\alpha|\Lambda|$ the overlap probability function restricted to this set should, in the limit of large Λ, coincide with the one of the whole system. Considering then the total overlap between two configurations $q = \alpha q_\Omega + (1-\alpha) q_{\Lambda-\Omega}$, it is clear that from the fact that q_Ω^2 and $q_{\Lambda-\Omega}^2$ are distributed like q^2 it follows that the product $q_\Omega q_{\Lambda-\Omega}$ is also distributed like q^2. Analogously for all r and s positive integers, $q_\Omega^r q_{\Lambda-\Omega}^s$ must be distributed like q^{r+s}, implying that $P(q_\Omega | q_{\Lambda-\Omega}) = \delta(q_\Omega - q_{\Lambda-\Omega})$, i.e. homogeneity of the overlap in space.[2]

In order to detect the possibility of an ordering phase transitions one can then impose "overlap boundary conditions" in the two-replica system. Suppose to fix the overlaps $\tilde{p}_k$ to a given value $\tilde{p} \in Supp(P(q))$ for all the Ω_k belonging to a boundary region of thickness $\sim \gamma^{-1}$ around the border, and to let otherwise free the boundary conditions. Long-range order would correspond to the fact that the probability of having a deviation of the local overlap from the value $\tilde{p}$, in a box situated in the bulk of the system, is vanishing for diverging box size.

In the following, we will concentrate on estimates of

$$F_\Lambda[\{\tilde{p}_k\}] = -\frac{1}{\beta} E \log B_\Lambda[\{\tilde{p}_k\}; J], \tag{15}$$

which is the disorder-averaged free energy functional corresponding to $B_\Lambda[\{\tilde{p}_k\}; J]$, i.e., the free energy of two replicas with fixed overlap profile. We will see in the next sections that saddle point approximation can be used to obtain a small-γ expansion of the free energy functional.

The rationale behind the definition (15) is that we expect the fluctuations of the probability $P_\Lambda[\{\tilde{p}_k\}; J]$ to be much smaller than its typical value, as long as $P_\Lambda \sim \exp(-aL^\mu)$ for some $a, \mu > 0$, so that the expectation of the logarithm gives its typical value[3]. Thanks to the the same self-averaging argument, it is reasonable to expect that the free energy of the two-replica system with fixed overlap boundary conditions can be obtained just considering the minimum of $F_\Lambda[\{\tilde{p}_k\}]$ over all the overlap profiles $\tilde{p}_k$ satisfying the boundary conditions. This (unproven) self-averaging property will be implicitly assumed to hold in the rest of the paper.

4 Replica analysis

The free energy $F_\Lambda[\{\tilde{p}_k\}]$ can be evaluated through the replica method for a system of constrained copies, which shows in a natural way how $F_\Lambda[\{\tilde{p}_k\}]$ takes, in the large-volume and

[2] We thank J. Kurchan for discussions on this point.

[3] It is easy to see, by the means of the usual interpolation techniques, that the fluctuations of $\log P$ cannot be larger than $O(L^{d/2})$, which implies the desired self-averaging property for those profiles such that $\mu > d/2$.

small-γ limit, the form of a rate function in a large deviation problem. For the sake of briefness we will not reproduce the full calculations, and we will just indicate the main steps in the derivation and the final results. In the next section we will then prove that, modulo some error terms which we will estimate, the rate functional found via the replica method provides a rigorous free energy lower bound, as it happens in the case of the unconstrained model [1].

As usual, the replica method starts from the expression of $E(B_\Lambda[\{\tilde{p}_k\};J]^n)$ for integer n. Introducing the replicas $\sigma_i^{r,a}$ for $r=1,2$ and $a=1,...,n$ and the local replica matrices $Q_k^{ra,sb}=\left(\frac{\gamma}{\delta}\right)^d\sum_{i\in\Omega_k}\sigma_i^{r,a}\sigma_i^{s,b}$ satisfying the symmetry conditions $Q_k^{ra,sb}=Q_k^{sb,ra}$ and such that the diagonal elements are fixed to $Q_k^{ra,ra}=1$ and $Q_k^{1a,2a}=\tilde{p}_k$, the average of the n-th moment can be written as an integral over the elements $Q_k^{ra,sb}$ which are not fixed:

$$\begin{aligned} E(B_\Lambda[\{\tilde{p}_k\};J]^n) &= \int\mathcal{D}Q\exp\left(\frac{\beta^2}{4}\left(\frac{\delta}{\gamma}\right)^d\sum_{k,m}\delta^d\phi(\delta|k-m|)\sum_{a,b}^{1,n}\sum_{r,s}^{1,2}Q_k^{ra,sb}Q_m^{ra,sb}\right)\\ &\quad\times\sum_{\{\sigma\}}e^{\beta h\sum_{i,r,a}\sigma_i^{r,a}}1_{\{q_k(\sigma^{r,a},\sigma^{s,b})=Q_k^{ra,sb};\ \forall\ a,b,r,s,k\}}. \end{aligned}$$

This expression is simplified introducing a continuum notation. Let us rescale the system by a factor γ and let $x\in V\equiv[0,L\gamma]^d$. Define then $Q_x^{ra,sb}=Q_k^{ra,sb}$ for $x/\gamma\in\Omega_k$ and write

$$E(B_\Lambda[\{\tilde{p}_k\};J]^n) = \int\mathcal{D}Q\exp\left(-\beta F^{Kac}_{(n)}[Q,\{\tilde{p}_k\}]\right) \tag{16}$$

$$\begin{aligned} &\equiv \int\mathcal{D}Q\exp\left(-\frac{\beta^2}{8\gamma^d}\sum_{r,s,a,b}\int_V d^dx\,d^dy\,\phi(|x-y|)(Q_x^{ra,sb}-Q_y^{ra,sb})^2\right)\\ &\quad\times\exp\left(-\frac{\beta}{\gamma^d}\int_V d^dx\,F^{(n)}_{(\delta/\gamma)^d}[Q_x,\tilde{p}_x]\right) \end{aligned} \tag{17}$$

where $F^{(n)}_{(\delta/\gamma)^d}$ is the free energy functional which one finds, within the replica method, when one computes the free energy of two Sherrington-Kirkpatrick (SK) systems with $(\delta/\gamma)^d$ spins, constrained to have a mutual overlap $\tilde{p}_x$. Note that $F^{(n)}$ has just a local dependence on the "overlap matrix" Q_x and on the overlap profile $\tilde{p}_x$. For large δ/γ a saddle point approximation can be considered and one needs an Ansatz for the matrix form in order to compute the $n\to 0$ continuation. This has been considered in [15, 16, 20] where the problem was treated assuming that each $n\times n$ matrix $\{Q_x^{ra,sb}\}_{1\le a,b\le n}$, for fixed indexes r,s, has a Parisi-like structure [3], and is therefore parametrized, in the limit $n\to 0$, by a functional order parameter $q_x^{r,s}(.):[0,1]\longrightarrow[0,1]$, verifying the symmetry $q_x^{r,s}(u)=q_x^{s,r}(u)$ and the monotonicity condition that for $0\le v\le u\le 1$, the 2×2 matrix in the indexes r and s, $\{q_x^{r,s}(u)-q_x^{r,s}(v)\}_{1\le r,s\le 2}$ are non-negative definite. One therefore needs to consider two families of functions $q_x(u)=q_x^{1,1}(u)=q_x^{2,2}(u)$ and $p_x(u)=q_x^{1,2}(u)=q_x^{2,1}(u)$, with $x\in V$ and $u\in[0,1]$.

4.1 Analytic continuation

In order to perform the analytic continuation of $F_{(n)}^{Kac}$ as $n \to 0$, it is useful to define the convolution of the functions $q_x^{r,s}(u)$ with the interaction potential

$$\hat{q}_x^{r,s}(u) = \int_V d^d y\, \phi(|x-y|) q_y^{r,s}(u), \tag{18}$$

and the Lagrange multiplier ϵ_x associated to the constrained local overlaps $\tilde{p}_x$. After some algebra one can write for the $n \to 0$ limit:

$$-\beta\gamma^d F_\Lambda^{Kac}[\{\tilde{p}_x\},\{q_x^{r,s}(.)\},\{\epsilon_x\}] = (2\log 2)|V| - \frac{\beta^2}{2}\int_V d^d x \left(1+\tilde{p}_x^2 - \int_0^1 du\, \left(q_x(u)^2 + p_x(u)^2\right)\right) \tag{19}$$

$$+\frac{\beta^2}{4}\int_{V\times V} d^d x\, d^d y \phi(|x-y|)\left((\tilde{p}_x-\tilde{p}_y)^2 - \int_0^1 du\, [(q_x(u)-q_y(u))^2 + (p_x(u)-p_y(u))^2]\right) \tag{20}$$

$$+\int_V d^d x\, (-\epsilon_x \tilde{p}_x + \log\cosh(\epsilon_x) + g_x(0,h,h;\epsilon_x))\,, \tag{21}$$

where $g_x(u,y_1,y_2;\epsilon_x)$, with $y_1, y_2 \in \mathbb{R}$, $u \in [0,1]$, is the solution of the backward parabolic equation[4]

$$\frac{\partial g_x}{\partial u} = -\frac{1}{2}\sum_{r,s}^{1,2} \frac{\partial \hat{q}_x^{rs}}{\partial u}\left(\frac{\partial^2 g_x}{\partial y_r \partial y_s} + u\frac{\partial g_x}{\partial y_r}\frac{\partial g_x}{\partial y_s}\right) \tag{22}$$

with final conditions

$$g_x(1,y_1,y_2;\epsilon_x) = \log\left[\cosh(\beta y_1)\cosh(\beta y_2)\right] + \log\left(1+\tanh(\epsilon_x)\tanh(\beta y_1)\tanh(\beta y_2)\right). \tag{23}$$

The functions $q_x^{r,s}(.)$ and ϵ_x are variational parameters over which one has to optimize to find the desired free energy (15) as a function of the overlap profile.

It is instructive to compare the Kac functional with the formula of the Parisi mean field free energy which, once optimized over $q(.)$, gives the free energy per spin of the SK model:

$$-\beta\mathcal{F}^{Parisi}[q(.)] = \log 2 - \frac{\beta^2}{4}\left(1 - \int_0^1 du\, q(u)^2\right) + f(0,h), \tag{24}$$

$f(u,y)$ being the solution of

$$\begin{cases} \frac{\partial f}{\partial u} = -\frac{1}{2}\frac{dq(u)}{du}\left(\frac{\partial^2 f}{\partial y^2} + u\left(\frac{\partial f}{\partial y}\right)^2\right) \\ f(1,y) = \log\cosh(\beta y) \end{cases}. \tag{25}$$

[4] One often considers the case where the functions $q_x^{r,s}(.)$ are piecewise constant. In this case, Eq. (22) has to be interpreted correctly, see Section 5

4.2 The p-spin case

We give here without proof the expression of the Kac free-energy functional for the p-spin model:

$$
\begin{aligned}
-\beta\gamma^d F_\Lambda^{Kac,p}[\{\tilde p_k\},\{q_k^{r,s}(.)\},\{\epsilon_k\}] = 2\log 2|V| - \frac{\beta^2(p-1)}{2}\int_V d^d x_1\cdots d^d x_p\,\phi(x_1,\cdots,x_p) \\
\times\Big[1+\tilde p_{x_1}\cdots\tilde p_{x_p} - \int_0^1 du\,\big(q_{x_1}(u)\cdots q_{x_p}(u)+p_{x_1}(u)\cdots p_{x_p}(u)\big)\Big] \\
+\int_V d^d x\,(-\epsilon_x\tilde p_x+\log\cosh(\epsilon_x)+\bar g_x(0,h,h;\epsilon_x))\,, \quad (26)
\end{aligned}
$$

where

$$\phi(x_1,\cdots,x_p)=\int_{\mathbb{R}^d} d^d k\,\psi(|x_1-k|)\cdots\psi(|x_p-k|)$$

and $\bar g_x(u,y_1,y_2;\epsilon_x)$ is the solution of the backward equation (22), with $\hat q_x^{r,s}(u)$ replaced by

$$\frac{p}{2}\int_V d^d x_2\cdots d^d x_p\,\phi(x,x_2,\cdots,x_p)q_{x_2}^{r,s}(u)\cdots q_{x_p}^{r,s}(u)$$

and with the same final condition (23).

5 Interpolating estimates

In this section show how the free energy functional (21) arises, without using the replica formalism. We start by fixing more precisely notations and definitions. For any box Ω_k, let $q_k^{r,s}(u)$, $r,s=1,2$ be functions

$$q_k^{r,s}(.):[0,1]\longrightarrow[0,1]$$

satisfying the following conditions:

- symmetry:
$$q_k^{1,1}(u)=q_k^{2,2}(u)\equiv q_k(u),\;\; q_k^{1,2}(u)=q_k^{2,1}(u)\equiv p_k(u) \quad (27)$$

- boundary values:
$$q_k^{r,s}(0)=0,\;\; q_k(1)=1,\;\; p_k(1)=\tilde p_k \quad (28)$$

- positive definiteness:

$$\text{for any } 0\le v\le u\le 1 \text{ the matrix } \{q_k^{r,s}(u)-q_k^{r,s}(v)\}_{r,s} \text{ is non-negative definite.} \quad (29)$$

Then,

Theorem 1. *For any choice of* $\{q_k^{r,s}(.)\}_k$ *satisfying conditions (27)-(29) and for any* $\{\epsilon_k\}$, *one has*

$$-\beta F_\Lambda[\{\tilde p_k\}] \le -\beta F_\Lambda^{Kac}[\{\tilde p_k\},\{q_k^{r,s}(.)\},\{\epsilon_k\}] + O(\delta)|\Lambda|. \tag{30}$$

One should keep in mind that δ is the size of the boxes, in the rescaled units, which has to be thought of as very small.

Remark We expect that also a lower bound of the type (30) holds. To prove this, a suitable generalization of Talagrand's theorem [2] would be needed.

Proof of Theorem 1 We introduce a generalization of the Aizenman, Sims and Starr's Random Overlap Structure [18], suitable to study a problem of two coupled replicas with spatially inhomogeneous structure. We introduce a set of possibly random weights $\xi_\alpha \ge 0$, where α takes values in a discrete set of indexes, such that

$$\sum_\alpha \xi_\alpha = 1,$$

and cavity fields $h_i^{\alpha,s}$, $\kappa^{\alpha,s}$, $i \in \Lambda$, and $s = 1,2$. The cavity fields are centered Gaussian random variables, independent of the couplings J_{ij}, with covariances

$$\begin{aligned} E(h_i^{\alpha,s}h_j^{\beta,r}) &= \delta_{ij}\sum_{m=1}^{(L\gamma/\delta)^d}\delta^d\phi(\delta|k-m|)q_m^{\alpha,s;\beta,r} && (31)\\ E(\kappa^{\alpha,s}\kappa^{\beta,r}) &= \frac{\delta^{2d}}{2\gamma^d}\sum_{k,m=1}^{(L\gamma/\delta)^d}\phi(\delta|k-m|)q_k^{\alpha,s;\beta,r}q_m^{\alpha,s;\beta,r}, && (32)\\ E(\kappa^{\alpha,s}h_i^{\beta,r}) &= 0 && (33)\end{aligned}$$

for $i \in \Omega_k$. We fix $q_k^{\alpha,1;\alpha,1} = q_k^{\alpha,2;\alpha,2} = 1$ and $q_k^{\alpha,1;\alpha,2} = \tilde p_k$, the constrained overlap in the box Ω_k, while all the other parameters are free (apart from the obvious constraint that the above covariance matrices are non-negative definite) and can be optimized to saturate the bounds. Notice that the cavity fields in different replicas are correlated. Define

$$H_t^\alpha(\sigma^1,\sigma^2) = \sqrt{t}\sum_{s=1}^{2}(H_\Lambda(\sigma^s,h{=}0;J)-\kappa^{\alpha,s}) - \sqrt{1-t}\sum_{i,s}h_i^{\alpha,s}\sigma_i^s - h\sum_{i,s}\sigma_i^s,$$

$Z_{\alpha,t} = Z_{\alpha,t}[\{\tilde p_k\}]$ as the respective partition function with constrained overlap profile $\{\tilde p_k\}$, Z_t as

$$Z_t = \sum_\alpha \xi_\alpha Z_{\alpha,t},$$

and the interpolating free energy $\mathcal{F}_t$ as

$$-\beta\mathcal{F}_t = E\log\frac{\sum_\alpha \xi_\alpha Z_{\alpha,t}}{\sum_\alpha \xi_\alpha e^{\beta\sum_s \kappa^{\alpha,s}}}. \tag{34}$$

For the t-derivative, one gets

$$
\begin{aligned}
-\beta\frac{\partial \mathcal{F}_t}{\partial t} &= \frac{\beta^2}{4} E Z_t^{-1}\sum_{\alpha,r,s}\xi_\alpha Z_{\alpha,t}\Bigg(\sum_{i,j\in\Lambda}\gamma^d\phi(\gamma|i-j|)\omega_\alpha(\sigma_i^r\sigma_i^s\sigma_j^r\sigma_j^s) \\
&\quad +2E(\kappa^{\alpha,s}\kappa^{\alpha,r}) - 2\sum_i E(h_i^{\alpha,s}h_i^{\alpha,r})\omega_\alpha(\sigma_i^r\sigma_i^s)\Bigg) \\
&\quad -\frac{\beta^2}{4} E Z_t^{-2}\sum_{\alpha,\beta,r,s}\xi_\alpha\xi_\beta Z_{\alpha,t}Z_{\beta,t}\Bigg(\sum_{i,j\in\Lambda}\gamma^d\phi(\gamma|i-j|)\omega^{(2)}_{\alpha,\beta}(\sigma_i^{r,1}\sigma_i^{s,2}\sigma_j^{r,1}\sigma_j^{s,2}) \\
&\quad +2E(\kappa^{\alpha,s}\kappa^{\beta,r}) - 2\sum_i E(h_i^{\alpha,s}h_i^{\beta,r})\omega^{(2)}_{\alpha,\beta}(\sigma_i^{r,1}\sigma_i^{s,2})\Bigg).
\end{aligned}
\tag{35}
$$

Here, $\omega_\alpha(.)$ denotes the Gibbs average, corresponding to the Hamiltonian $H_t^\alpha(\sigma^1,\sigma^2)$, acting on the two replicas σ^1,σ^2 with constrained overlap profile. On the other hand, $\omega^{(2)}_{\alpha,\beta}$ is the *duplicated* Gibbs average, involving the *four* replicas $\sigma^{r,s}, r,s=1,2$, with Hamiltonian

$$H_t^\alpha(\sigma^{1,1},\sigma^{2,1}) + H_t^\beta(\sigma^{1,2},\sigma^{1,2}).$$

Note that, in the average $\omega^{(2)}_{\alpha,\beta}$, only the overlap profiles between replicas $\sigma^{1,1},\sigma^{2,1}$ and between $\sigma^{1,2},\sigma^{2,2}$ are constrained to $\{\tilde p_k\}$. Thanks to the constraints on the local overlap profile and to the choice for the covariances of the cavity fields, the first term in the derivative is at most of order $O(\delta)|\Lambda|$ (it is not exactly zero, since ϕ has small variations within each box). The second one gives

$$
\begin{aligned}
&-\frac{\beta^2}{4}\left(\frac{\delta}{\gamma}\right)^{2d} E\Bigg\{ Z_t^{-2}\sum_{\alpha,\beta,r,s}\xi_\alpha\xi_\beta Z_{\alpha,t}Z_{\beta,t}\sum_{k,m}\gamma^d\phi(\delta|k-m|) \\
&\times\,\omega^{(2)}_{\alpha,\beta}[(q_k(\sigma^{r,1},\sigma^{s,2})-q_k^{\alpha,s;\beta,r}))(q_m(\sigma^{r,1},\sigma^{s,2})-q_m^{\alpha,s;\beta,r}))]\Bigg\} + O(\delta)|\Lambda|.
\end{aligned}
\tag{36}
$$

The average is negative since ϕ is non-negative definite by the assumption (4), so that we get the bound

$$-\beta F_\Lambda[\{\tilde p_k\}] = -\beta\mathcal{F}_1 \le -\beta\mathcal{F}_0 + O(\delta)|\Lambda|. \tag{37}$$

To complete the proof of Eq. (30), we have to show that there exists a choice of the ξ_α and of the cavity fields such that $\mathcal{F}_0$ coincides with $F_\Lambda^{Kac}[\{\tilde p_k\},\{q_k^{r,s}(.)\},\{\epsilon_k\}]$. As usual, it is sufficient to consider the case where the functions $q_k^{r,s}(.)$ are piecewise constant, the general case being obtained as a limit. Let $K\ge 1$, $0=m_0\le m_1\le\cdots\le m_K=1$ and define

$$q_k^{r,s}(u) = q_k^{r,s}[0] \tag{38}$$

$$q_k^{r,s}(u) = q_k^{r,s}[\ell+1] \quad \text{for} \quad m_\ell \le u < m_{\ell+1} \tag{39}$$

$$q_k^{r,s}(1) = q_k^{r,s}[K+1], \tag{40}$$

where $q_k^{r,s}[\ell]$, $\ell = 0, \cdots, K$ are parameters satisfying the analog of conditions (27)-(29):

$$q_k^{1,1}[\ell] = q_k^{2,2}[\ell] \equiv q_k[\ell], \quad q_k^{1,2}[\ell] = q_k^{2,1}[\ell] \equiv p_k[\ell],$$

$$q_k^{r,s}[0] = 0, \quad q_k[K+1] = 1, \quad p_k[K+1] = \tilde{p}_k$$

and

$$\{q_k^{r,s}[\ell] - q_k^{r,s}[\ell-1]\}_{r,s} \quad \text{non-negative definite.} \tag{41}$$

If $\alpha = (\alpha_1, \cdots, \alpha_K)$ with $\alpha_\ell \in \mathbb{N}$, we define the cavity fields as

$$\kappa^{\alpha,s} = y^{(0,s)} + y^{(1,s)}_{\alpha_1} + \cdots + y^{(K,s)}_{\alpha_1,\cdots,\alpha_K} \tag{42}$$

and

$$h_i^{\alpha,s} = z^{(i,0,s)} + z^{(i,1,s)}_{\alpha_1} + \cdots + z^{(i,K,s)}_{\alpha_1,\cdots,\alpha_K}, \tag{43}$$

where the y's and the z's are centered Gaussian variables with covariances

$$\begin{aligned} E(y^{(\ell,s)}_{\alpha_1,\cdots,\alpha_\ell} y^{(\ell',r)}_{\beta_1,\cdots,\beta'_\ell}) &= \delta_{\ell\ell'}\delta_{\alpha\beta}\frac{\delta^{2d}}{2\gamma^d}\sum_{k,m}\phi(\delta|k-m|)\,\{q_k^{r,s}[\ell+1]q_m^{r,s}[\ell+1] - q_k^{r,s}[\ell]q_m^{r,s}[\ell]\} \qquad (44)\\ &= \frac{\delta_{\ell\ell'}\delta_{\alpha\beta}}{2}\int_{V\times V} d^dx\, d^dy\, \phi(|x-y|)\,\{q_x^{r,s}[\ell+1]q_y^{r,s}[\ell+1] - q_x^{r,s}[\ell]q_y^{r,s}[\ell]\} \end{aligned}$$

and

$$\begin{aligned} E(z^{(i,\ell,s)}_{\alpha_1,\cdots,\alpha_\ell} z^{(j,\ell',r)}_{\beta_1,\cdots,\beta'_\ell}) &= \delta_{ij}\delta_{\ell\ell'}\delta_{\alpha\beta}\sum_m \delta^d\phi(\delta|k-m|)\,\{q_m^{r,s}[\ell+1] - q_m^{r,s}[\ell]\} \qquad (45)\\ &= \delta_{ij}\delta_{\ell\ell'}\delta_{\alpha\beta}\,\{\hat{q}_x^{r,s}[\ell+1] - \hat{q}_x^{r,s}[\ell]\}\,, \end{aligned}$$

for $i \in \Omega_k$. This is equivalent to say that the parameters $q_k^{\alpha,s;\beta,r}$ in (31)-(32) are given by

$$q_k^{\alpha,s;\beta,r} = q_k^{r,s}[\ell] \quad \text{if} \quad \alpha_a = \beta_a \quad \text{for} \quad a < \ell \quad \text{and} \quad \alpha_\ell \neq \beta_\ell. \tag{46}$$

Note that the covariance matrix of the z's is well defined thanks to condition (41). As for the variables y's, the same is true thanks to Lemma 1 in Appendix. As for the random weights ξ_α, we choose them as in [18] to be the Ruelle probability Cascade with K levels and parameters $m_1, \cdots, m_K$.

At this point, using the properties of the Ruelle Probability Cascades (see for instance [21]), it is not difficult to compute explicitly $\mathcal{F}_0$. For the denominator, one finds

$$\gamma^d E\log\sum_\alpha \xi_\alpha e^{\beta\sum_s \kappa^{\alpha,s}} = \frac{\beta^2\delta^{2d}}{2}\sum_{k,m}\phi(\delta|k-m|)\left[1 + \tilde{p}_k\tilde{p}_m - \int_0^1 du\ [q_k(u)q_m(u) + p_k(u)p_m(u)]\right] \tag{47}$$

which, in the continuum notation, can be rewritten as

$$
\begin{aligned}
&\frac{\beta^2}{2}\int_V d^dx \left(1+\tilde{p}_x^2-\int_0^1 du\ \left(q_x(u)^2+p_x(u)^2\right)\right) \qquad (48)\\
&-\frac{\beta^2}{4}\int_{V\times V} d^dx\, d^dy\, \phi(|x-y|)\left((\tilde{p}_x-\tilde{p}_y)^2-\int_0^1 du\,[(q_x(u)-q_y(u))^2+(p_x(u)-p_y(u))^2]\right)\\
&+O(\delta)|V|.
\end{aligned}
$$

As for the numerator of $-\beta\mathcal{F}_0$, it is not difficult to see that it is bounded above by

$$
\begin{aligned}
&|\Lambda| 2\log 2-\gamma^{-d}\int_V d^dx\, \epsilon_x\tilde{p}_x\\
&\qquad + E\log\sum_\alpha \xi_\alpha \prod_k\prod_{i\in\Omega_k}\left[\cosh(\epsilon_k)\cosh(\beta h_i^{\alpha,1}+\beta h)\cosh(\beta h_i^{\alpha,2}+\beta h)\right.\\
&\qquad\qquad \left.+\sinh(\epsilon_k)\sinh(\beta h_i^{\alpha,1}+\beta h)\sinh(\beta h_i^{\alpha,2}+\beta h)\right], \qquad (49)
\end{aligned}
$$

where ϵ_k has the meaning of a Lagrange multiplier which implements the constraint $q_k(\sigma^1,\sigma^2)=\tilde{p}_k$. The bound holds for any choice of $\{\epsilon_k\}$, as it follows from the obvious inequality

$$
\sum_{q_k(\sigma^1,\sigma^2)=\tilde{p}_k} e^{\cdots}=\sum_{q_k(\sigma^1,\sigma^2)=\tilde{p}_k} e^{\cdots+\epsilon_k(q_k(\sigma^1,\sigma^2)-\tilde{p}_k)}\le\sum_{\sigma^1,\sigma^2} e^{\cdots+\epsilon_k(q_k(\sigma^1,\sigma^2)-\tilde{p}_k)}. \qquad (50)
$$

Again using the properties of the Ruelle Cascades, the expression (49) can be rewritten as

$$
(2\log 2+O(\delta))|\Lambda|+\gamma^{-d}\int_V d^dx\,(-\epsilon_x\tilde{p}_x+\log\cosh(\epsilon_x)+g_x(0,h,h;\epsilon_x)), \qquad (51)
$$

where $g_x(u,h_1,h_2;\epsilon_x)$ is the solution of the backward parabolic equation (22), with final conditions (23). □

At least when p is even, Theorem 1 can be immediately extended to the p-spin case.

6 Analysis of the rate function

The functionals (21) and (26) allow in principle to study the spin glass problem for small but finite γ. The difficulties one has to face in this respect are of two types. First of all, the analysis of the Kac functionals themselves is technically very involved, since for a given overlap profile one needs to consider a variational principle, whose solution cannot in general be found explicitly. On the other hand, the analysis of the Kac functional is not sufficient to infer the behavior of the probability of overlap profiles $P_\Lambda[\{\tilde{p}_k\};J]$. Indeed, a careful analysis of the error terms due to finite-volume and finite-γ effects is also necessary, as it is already in the ferromagnetic case [22].

In order to overcome the first problem, in the present section we restrict to a situation (temperature close to the critical temperature of mean field theory and overlap profiles sufficiently smooth to apply a gradient expansion) where the Kac functionals can be reasonably replaced by an approximate form, which allows for an explicit treatment of the optimization problem. The mathematical justification of the approximations involved, together with the study of the error terms, is left to future research.

6.1 Homogeneous solution

In order to find the optimal estimate for the free energy $F_\Lambda[\{\tilde{p}_k\}]$ of the two-replica system with constrained overlap profile, one has to minimize the Kac functional (21) with respect to the Parisi functions $q_x^{r,s}(.)$ and to the Lagrange multipliers ϵ_x. The variational problem for coupled replicas has an evident degeneracy, at least in the case of overlap profiles constant in space. In fact, suppose to take all the $\tilde{p}_k$ equal to a value $\tilde{p}$ in the support of the $P(q)$ function for the single-replica problem. Then, if the Parisi function $q_F(u)$ solves the variational problem in the case of a single replica, i.e., if it minimizes the expression (24), the following position-independent form of the functions $q_x(u)$ and $p_x(u)$ solves the two-replica variational problem:

$$q_x(u) = \begin{cases} q_F(2u) & u \le \tilde{u}/2 \\ \tilde{p} & \tilde{u}/2 < u \le \tilde{u} \\ q_F(u) & u > \tilde{u} \end{cases} \tag{52}$$

$$p_x(u) = \begin{cases} q_F(2u) & u \le \tilde{u}/2 \\ \tilde{p} & u > \tilde{u}/2 \end{cases}, \tag{53}$$

where $\tilde{u}$ is defined as the value of u for which $q_F(\tilde{u}) = \tilde{p}$. (It is immediate to verify that the definite positiveness condition (29) is satisfied). At the same time, the variational equations with respect to ϵ_x are solved by $\epsilon_x = 0$. It is then possible to see, comparing formulas (21) and (24), that with this choice for the variational parameters $q_x^{r,s}$ and ϵ_x, the Kac functional equals twice the free energy of the single-replica mean field system, and is therefore *independent of* $\tilde{p}$. Indeed, it is immediate to verify this property explicitly for the polynomial part (48) of the free energy, which reduces to

$$-\frac{\beta^2}{2}|V| \left(1 - \int_0^1 du\, q_F(u)^2\right). \tag{54}$$

As for the term (51) involving the parabolic equation, one can see that, if $f_F(u, y)$ is the solution of (25) with $q(u) = q_F(u)$, the following form solves the equation (22) with the choice (52)-(53):

$$g(u, y_1, y_2) = \begin{cases} 2f_F(2u, y_1)\delta(y_1 - y_2) & u \le \tilde{u}/2 \\ 2f_F(\tilde{u}, y_1)\delta(y_1 - y_2) & \tilde{u}/2 \le u \le \tilde{u} \\ f_F(u, y_1) + f_F(u, y_2) & u > \tilde{u} \end{cases}. \tag{55}$$

Similar considerations show also that (51) then equals twice the term one has in the free energy functional of the SK model.

Notice that in the spin glass $p = 2$ case, where at low temperature the support of the function $P(q)$ is an interval $[q_{min}, q_{max}]$, this degeneracy corresponds to the fact that the free energy of the two-replica system has a continuous zero mode (at least if finite-γ effects are neglected). In the p-spin case, on the other hand, the support at low temperature is given by two points $\{q_0, q_1\}$, and one has just a discrete degeneracy.

In both cases the degeneracy reflects the fact that a constant overlap overlap profile, whose value lies in the support of the mean field distribution function $P(q)$, cannot have too small a probability in typical samples, as it already follows from the results of [13]. More precisely: the probability under consideration could be in principle exponentially small with the system size[5], $P \sim \exp(-a(\gamma)L^d)$, but in that case $a(\gamma)$ has to vanish in the Kac limit $\gamma \to 0$.

6.2 The Parisi model close to T_c

Since the analysis of the complete functionals is technically very involved, already at the mean field level it is customary to resort to simplifying approximations that, while representing correctly the physics in some limiting cases, allow for an analytic study of the saddle point equations. In the case of $p = 2$, where the mean field spin glass transition is of second order, close to $T = T_c = 1$, $h = 0$ one can use a Landau expansion in the order parameter, assuming that the Parisi function $q_F(u)$ is close to zero for any $u < 1$. On the other hand for the p-spin case ($p > 2$), where the transition has a first order character with a discontinuous order parameter, one usually resorts to a spherical approximation of continuous spins with a global constraint.

The scope of this section is to begin the study of the large deviation functional for the Kac spin glass with pair interactions close to the mean field critical temperature. For the SK model, the Landau expansion truncated to the fourth order is a good representation of the free energy of the model in the vicinity if the critical temperature, accurate to the fifth order in $\tau = T_c - T$. Parisi proposed [23] to simplify the expansion, retaining among all the fourth order terms only the one responsible for replica symmetry breaking. It has been shown that this approximation does not affect the free energy to the fifth order in τ. In the Kac model the Landau expansion can be supplemented by a gradient expansion, which corresponds to the assumption that all the functions appearing in (21) have small variations in space on scales

[5]This does certainly happen for instance in dimension 1, where there is no phase transition and the overlap is peaked at a single value as long as γ is finite.

comparable to the interaction range γ^{-1}. The resulting free energy can be written as:

$$
\begin{aligned}
-\beta\gamma^d F_\Lambda^{Kac} \;=\; & -\frac{c^*}{2}\int d^dx\;(\nabla\tilde{p}_x)^2+\frac{\beta^2}{2}\int d^dx\;\tilde{p}_x^2+\frac{1}{2}\int d^dx\;\epsilon_x^2-\int d^dx\;\tilde{p}_x\epsilon_x \\
& -\frac{c}{2}\int dx\;[(\nabla\epsilon_x)^2-\int_0^1 du\;((\nabla\hat{q}_x(u))^2+(\nabla\hat{p}_x(u))^2)] \\
& +\frac{1-T^2}{2}\int dx\;[\epsilon_x^2-\int_0^1 du\;(\hat{q}_x(u)^2+\hat{p}_x(u)^2)] \\
& +\int d^dx\;\Big(\frac{1}{3}\big[2\langle\hat{q}_x\rangle\langle\hat{q}_x^2\rangle+\int_0^1 du\;\hat{q}_x(u)\int_0^u dv(\hat{q}_x(u)-\hat{q}_x(v))^2+6\langle\hat{p}_x\hat{q}_x\rangle(\langle\hat{p}_x\rangle-\epsilon_x) \\
& +3\int_0^1 du\,\hat{q}_x(u)\int_0^u dv\;(\hat{p}_x(u)-\hat{p}_x(v))^2\big] \\
& +\frac{1}{6}[\epsilon_x^4-\int_0^1 du\;(\hat{q}_x(u)^4+\hat{p}_x(u)^4)]\Big) \\
& +h^2\int d^dx\;[\epsilon_x-\int_0^1 du\;(\hat{q}_x(u)+\hat{p}_x(u))] \qquad (56)
\end{aligned}
$$

where $c=\frac{T^2}{2}\int d^dx\,\phi^{-1}(|x|)x^2$, $c^*=\beta^4 c$ and we have introduced the notation $\langle\cdot\rangle=\int_0^1 du\;\cdot(u)$. In performing the expansion, we have assumed that $q_x(u)$ and $p_x(u)$ are of order τ and we have subtracted an inessential constant factor, corresponding to the unconstrained free energy of the mean-field single-replica model.

Instead of trying to justify the expansion (56) we will take it as a *fait accompli* and use it for our preliminary study of the rate function. Of course a full justification would involve the proof that the neglected terms are much smaller than the retained one. We expect this to be harmlessly true for $T>T_c$, while care should be taken in the discussion of the case $T<T_c$.

6.2.1 $T>T_c$

In this section we discuss the interface problem for temperatures higher than T_c, i.e., for $\tau<0$. In these conditions the mean field system is paramagnetic and for $\gamma\to 0$ the only values of $\tilde{p}_k$ giving the same free energy as the unconstrained solution are given by $\tilde{p}_k=0$, and correspondingly $q_k(u)=p_k(u)=0$. We would like now to impose an overlap value $\tilde{p}_k=\tilde{p}_0$ in the boundary and study the decay of the overlap to zero in the bulk, and the free energy cost for the boundary conditions. For simplicity we will suppose that the overlap boundary conditions are imposed on a given $d-1$ dimensional hyperplane, so that we can limit ourselves to a one-dimensional problem for the transverse direction. In other words, the boundary conditions will be $\tilde{p}_k=\tilde{p}_0$ for all k such that $k_1=0$, and periodic in the other directions. We will look for replica symmetric solutions to the saddle point equations (which amounts to assume that $q_k(u)=q_k$ and $p_k(u)=p_k$ are constant for $0\le u<1$) and use a continuum limit formulation.

The reasons for the replica-symmetric choice (apart from its simplicity) will be briefly discussed in the following. The one-dimensional free-energy density one obtains from (56) is then

$$\begin{aligned}
-\beta\gamma^d f_x^{Kac} &= \frac{-c^*}{2}(\partial_x\tilde{p}_x)^2+\frac{\beta^2}{2}\tilde{p}_x^2+\frac{1}{2}\epsilon_x^2-\tilde{p}_x\epsilon_x-\frac{c}{2}[(\partial_x\epsilon_x)^2-(\partial_x\hat{q}_x)^2-(\partial_x\hat{p}_x)^2]\\
&+\tau(\epsilon_x^2-\hat{q}_x^2-\hat{p}_x^2)+\frac{1}{3}[2\hat{q}_x^3+6\hat{p}_x\hat{q}_x(\hat{p}_x-\epsilon_x)]\\
&+\frac{1}{6}[\epsilon_x^4-\hat{q}_x^4-\hat{p}_x^4]+h^2(\epsilon_x-\hat{p}_x-\hat{q}_x).
\end{aligned} \tag{57}$$

"Equations of motion" are obtained by requiring the action to be stationary:

$$\begin{aligned}
&c^*\partial_x^2\tilde{p}_x+\beta^2\tilde{p}_x-\epsilon_x=0\\
&c\partial_x^2\epsilon_x+\epsilon_x-\tilde{p}_x+2\tau\epsilon_x-2\hat{p}_x\hat{q}_x+\frac{2}{3}\epsilon_x^3+h^2=0\\
&c\partial_x^2\hat{q}_x+2\tau\hat{q}_x-2\hat{p}_x^2+2\hat{p}_x\epsilon_x+\frac{2}{3}\hat{q}_x^3+h^2=0\\
&c\partial_x^2\hat{p}_x+2\tau\hat{p}_x-4\hat{q}_x\hat{p}_x+2\hat{q}_x\epsilon_x+\frac{2}{3}\hat{p}_x^3+h^2=0.
\end{aligned} \tag{58}$$

The analysis of these equations for negative τ is straightforward. As an illustration we discuss the case $h=0$, the more general case being quite similar. For $h=0$ the equations admit a solution $\hat{q}_x=\hat{p}_x=0$ and

$$\begin{aligned}
&c^*\partial_x^2\tilde{p}_x+\beta^2\tilde{p}_x-\epsilon_x=0\\
&c\partial_x^2\epsilon_x+\epsilon_x-\tilde{p}_x+2\tau\epsilon_x+\frac{2}{3}\epsilon_x^3=0
\end{aligned} \tag{59}$$

Since we are assuming that $\tilde{p}_x=O(\tau)$, the cubic term in (59) can be neglected to the leading order, and one has the solution

$$\tilde{p}_x=\epsilon_x=\tilde{p}_0e^{-\sqrt{2|\tau|/c}\,x} \tag{60}$$

which shows that the memory of the boundary condition is lost exponentially fast. Note that the solution is actually slowly varying, which means that the gradient expansion is self-consistent. Other solutions of (59) exist (the boundary conditions for ϵ_x are not specified) but they are no longer slowly varying or diverge for $x\to\infty$. These have to be discarded, since in that case the approximations involved in (56) clearly break down.

Inserting it in the free energy and integrating over space one finds

$$-\beta\gamma^dF_\Lambda^{Kac}[\tilde{p}_0]=-\sqrt{2c|\tau|}\tilde{p}_0^2(\gamma L)^{d-1}+O(\tilde{p}_0^4/\sqrt{|\tau|})(\gamma L)^{d-1}, \tag{61}$$

which implies that the boundary conditions imposed require a surface free energy cost

$$\Delta F\simeq\frac{L^{d-1}}{\gamma}\tilde{p}_0^2\sqrt{2c|\tau|}. \tag{62}$$

for the imposed boundary conditions. Recall that this result has been obtained within a replica-symmetric Ansatz for the Parisi functions. Thanks to the variational character of Theorem 1, this implies that the free energy cost for the chosen boundary condition is *not smaller* than (62). Note the appearance of a divergent length $\xi \sim |\tau|^{-1/2}$ as the transition is approached. This is the mean field exponent for the correlation length, which does not take into account renormalization effects.

6.2.2 $T < T_c$

The case $T < T_c$ is the relevant one to study the possibility of long-range order of Parisi type. Ideally one would like to prove (or disprove) the existence of a finite critical dimension d_{LC} above which long-range order is present for small enough γ. As we stressed in the introduction, the issue can be formulated as a problem of sensitiveness to overlap boundary conditions for coupled-replica systems. The knowledge of the exact large deviation functional, the free-energy cost as a function of the overlap profile, would in principle allow to reduce this problem to the proof of the existence of a phase transition in a (non-random) field-theoretical model. Our Theorem 1 provides a lower bound to the large deviation functional, thus if we define d^*_{LC} as the lower critical dimension for the approximated theory, one can argue that $d_{LC} \leq d^*_{LC}$[6]. In order to study the stability of long-range order, one should identify the relevant excitations around the homogeneous solution (52)-(53), that induce overlaps different from $\tilde{p}_0$ in the center of the system, $\tilde{p}_0 \in Supp(P(q))$ being the chosen overlap boundary condition. In the following of this section we will argue that excitations localized in a region of size ℓ have a free-energy cost of the order ℓ^{d-a}/γ^a. On the other hand, in analogy with the usual Peierls argument for the Ising model, one should also find a way to estimate the number of such excitations and show that the corresponding entropic gain does not overcome the free-energy cost at low enough γ. Let us for example imagine that the number of excitations grows with ℓ as $\exp(c\ell^{d-b})$. It is clear that if $b = a$, as it happens in models without disorder, then for γ small enough and $d > a$ large excitations will be exponentially suppressed and $d^*_{LC} = a$. Conversely, if $b < a$ the entropy terms will dominate for all d and destroy the phase transition for all finite γ.

An equivalent method to determine long-range order makes use of "twisted" boundary conditions. In that case one direction is singled out and, while periodic conditions are assumed in the remaining directions, $\tilde{p}_1$ and $\tilde{p}_2$ conditions are chosen at the boundaries along the preferred direction [19]. Again, in a system of linear size L one can expect profiles with free-energy cost proportional to L^{d-a}/γ^a and an entropy proportional to $(\gamma L)^{d-b}$.

The "energetic" contribution can be estimated for small γ as the value of the large-deviation functional corresponding to the minimizing profile. The study of non-uniform saddle points of the free-energy (56) corresponding to inhomogeneous boundary conditions to determine the cost for overlap interfaces have been first put forward in [19]. In that paper the saddle point procedure was assumed without further justification, while here it legitimated by the appearance

[6]Of course, one should also show that the "error terms" $O(\delta)$ in Theorem 1 are not dangerous

of the interaction volume γ^{-d} in front of the action. The results of [19] can be mutuated directly to our case.

In that paper it was assumed that the solution of the optimization problem could be obtained by a perturbation of the form (52)-(53), and it was found that the form (52)-(53) should be modified for values of u around $\tilde{u}/2$ and $\tilde{u}$. Thanks to this modification, and translating in terms of the Kac model, the scaling of the free energy results to be of the order $L^{d-5/2}/\gamma^{5/2}$, thus modifying the values of the exponent a from 3 to 5/2. This again should be considered as a (better) upper bound estimation, and if we suppose that this is the exact value, we conclude as in [19] that "energetic" effects destroy the possibility of Parisi order in dimension $d = 1, 2$, while they are consistent with that kind of order in dimension 3 and above.

We would like to conclude by remarking that a simple estimate making use of the "homogeneous solution" (52)-(53) does not apply to the p-spin model, and more generally to models with one-step RSB. In these cases the function $q_F(u)$ has the form $q_F(u) = \theta(m-u)q_0 + \theta(u-m)q_1$ and the support of $P(q)$ is just the set $\{q_0, q_1\}$. There are only two homogeneous solutions of the kind (52)-(53) corresponding to the two possible costless overlap values for $\tilde{p}$, q_0 and q_1. The analogous of the hypothesis of neglecting the kinetic contribution in the saddle point equations would consist in assuming in different points of space either one or the other possible form. But in this case, it is simple to realize that the kinetic term would be identically equal to zero. In order to get an estimate of the free-energy cost for an interface, a detailed solution of the space-dependent saddle point equations is necessary [24].

7 Conclusions

The main result of this paper is the derivation of large deviation functional, quantifying for small γ the probability of overlap profiles in spin glass models with Kac-type interactions. We obtain that the replica expression provides a lower bound to the true free-energy functional. Moreover, putting aside the problem of mathematically justifying the approximations involved, we performed a first analysis of the free-energy functional, finding estimates for the free-energy cost of extended "overlap interfaces".

In this paper we have considered the case of two replicas coupled in a symmetric way. Although we did not discuss it, the generalization of our analysis to R replicas, which can be useful to discuss issues related to ultrametricity [16], can be achieved in a rather straightforward way. Another possible and more interesting generalization concerns the introduction of a "quenched potential" where again one considers two replicas, but coupled in an asymmetric way. One considers a first replica in an arbitrary configuration chosen with the usual Boltzmann weight, and then a second one, constrained to have a given overlap profile with the first [20]. This approach, more involved than the one presented here, would be especially relevant for the case of the p-spin model, where one would like to study nucleation of entropic droplets in a given equilibrium state [25, 26].

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A Covariance of the cavity fields $y_\alpha^{(\ell,s)}$

Let $\ell = 0, \cdots, K$, $r = 1, 2$, and

$$\hat{A}_\ell^{(r)} = \begin{pmatrix} a_{11}^{(r)}(\ell) & a_{12}^{(r)}(\ell) \\ a_{12}^{(r)}(\ell) & a_{11}^{(r)}(\ell) \end{pmatrix}. \tag{63}$$

The positive-definiteness of the covariance of the variables $y_\alpha^{(\ell,s)}$ in Section 5 follows from the following Lemma:

Lemma 1. *Assume that* $\hat{A}_\ell^{(r)} - \hat{A}_{\ell-1}^{(r)}$, $\ell = 1, \cdots, K$, $r = 1, 2$ *are non-negative definite. Then, the same holds for* $\hat{B}_\ell - \hat{B}_{\ell-1}$, *where*

$$\hat{B}_\ell = \begin{pmatrix} a_{11}^{(1)}(\ell)a_{11}^{(2)}(\ell) & a_{12}^{(1)}(\ell)a_{12b}^{(2)}(\ell) \\ a_{12}^{(1)}(\ell)a_{12}^{(2)}(\ell) & a_{11}^{(1)}(\ell)a_{11}^{(2)}(\ell) \end{pmatrix} \tag{64}$$

Proof of Lemma 1 From the hypothesis follows that

$$a_{11}^{(r)}(\ell) \pm a_{12}^{(r)}(\ell) \geq 0 \tag{65}$$

$$\Delta_{11}^{(r)} \pm \Delta_{12}^{(r)} \geq 0, \tag{66}$$

where $\Delta_{cd}^{(r)} = a_{cd}^{(r)}(\ell) - a_{cd}^{(r)}(\ell-1)$. The statement of the Lemma is equivalent to the non-negativity of

$$[a_{11}^{(1)}(\ell)a_{11}^{(2)}(\ell) - a_{11}^{(1)}(\ell-1)a_{11}^{(2)}(\ell-1)] \pm [a_{12}^{(1)}(\ell)a_{12}^{(2)}(\ell) - a_{12}^{(1)}(\ell-1)a_{12}^{(2)}(\ell-1)], \tag{67}$$

which follows immediately from Eqs. (65)-(66), once (67) is rewritten as

$$\begin{aligned} &(\Delta_{11}^{(1)} + \Delta_{12}^{(1)})\frac{a_{11}^{(2)}(\ell) \pm a_{12}^{(2)}(\ell)}{2} + (\Delta_{11}^{(1)} - \Delta_{12}^{(1)})\frac{a_{11}^{(2)}(\ell) \mp a_{12}^{(2)}(\ell)}{2} \\ &+(\Delta_{11}^{(2)} + \Delta_{12}^{(2)})\frac{a_{11}^{(1)}(\ell-1) \pm a_{12}^{(1)}(\ell-1)}{2} + (\Delta_{11}^{(2)} - \Delta_{12}^{(2)})\frac{a_{11}^{(1)}(\ell-1) \mp a_{12}^{(1)}(\ell-1)}{2}. \end{aligned} \tag{68}$$

□

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